AMO
InfoPack
2021

American
Mathematics Olympiad

Internal Code SIUC–STEM–STS–01–A
Edition 1.0

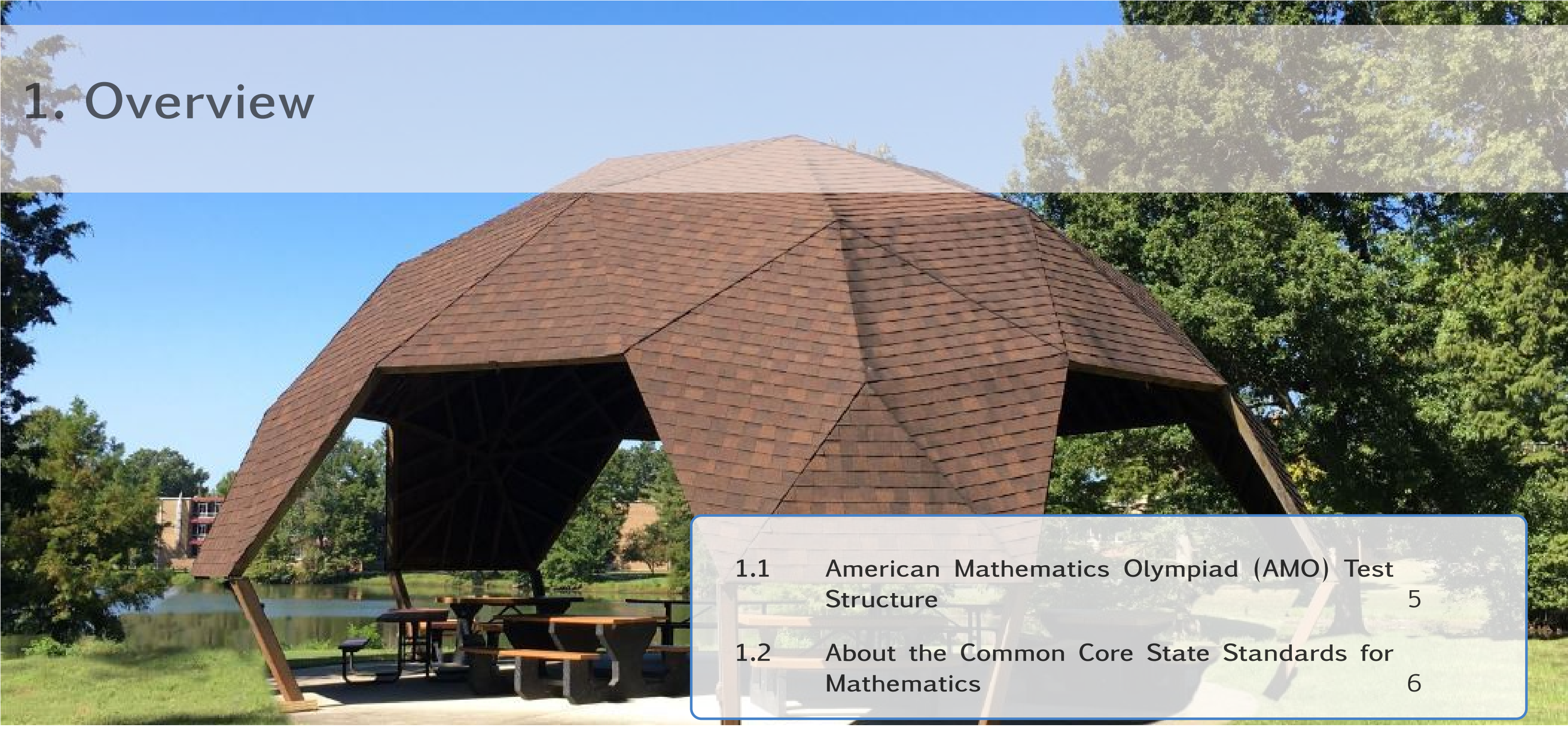
Cover Design by SIUC STEM EDUCATION RESEARCH CENTER

Published by Southern Illinois University Carbondale in Collaboration with SIMCC and Scholastic Trust Singapore as the American Mathematics Olympiad (AMO).

Designed in Carbondale, Illinois, USA.
LaTeXTypesetting Based on a Textbook Template from Typsetters.Se

Contents

1. Overview



1.1 American Mathematics Olympiad (AMO) Test Structure

Grades 2-11 and 12		
Section A 15 MC Questions (3 points each) Total: 45 points	Section B 5 Open-Ended Questions (5 points each) Total: 25 points	Section C 5 Open-Ended Questions (6 points each) Total: 30 points
Total: 100 points (no calculators allowed) (no points deducted from wrong responses)		

Overview of the American Mathematics Olympiad (AMO) Test Structure

1.2 About the Common Core State Standards for Mathematics

In developing the current AMO test items, the Common Core State Standards for Mathematics[1] (CCSSM, 2010) are consulted and extended in the context of AMO mathematics competition and its international audience. The articulations and extensions do not necessarily represent the views of the Authors of the CCSSM standards and are solely those of the AMO team for the purposes of mathematical engagement and enrichment of young learners.

[1] National Governors Association Center for Best Practices and Council of Chief State School Officers. (2010). *Common Core State Standards for Mathematics*. Washington, DC: Authors. Available at http://www.corestandards.org/Math/

2. Grade Two

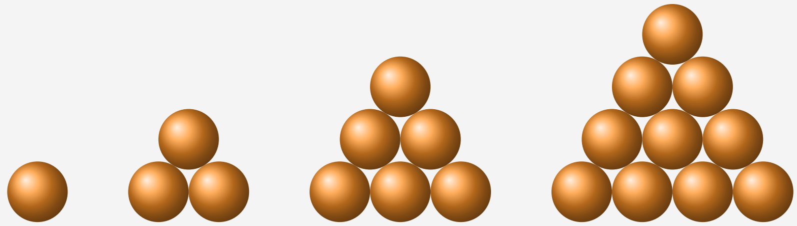
In the second grade, the test covers basic mathematical ideas related to the base-ten number system, addition and subtraction, number patterns, measurements in the real world, and simple 2D shapes and their properties. Six examples are listed with explanations.

2.1 Targeted Mathematical Ideas

- Place values in the base-ten number system.
- Simple fractions such as halves, thirds, and quarters or fourths.
- Addition and subtraction and related multiplication and division problems involving numbers less than 100.
- Simple number patterns involving addition or subtraction.
- Measuring length, time, and money, and collecting data in real-world contexts.
- Relating numbers less than 100 to points on the number lines.
- Basic counting strategies and reasoning in the context of life-like stories.
- Recognize, count, and see connections among simple 2D shapes such as triangles and rectangles.
- Problem solving strategies in the above areas.

2.2 Examples and Solutions

Example 2.1 The following shapes of oranges go on forever from the left to the right. How many oranges are there in the fifth shape?



(A) 12

(B) 15

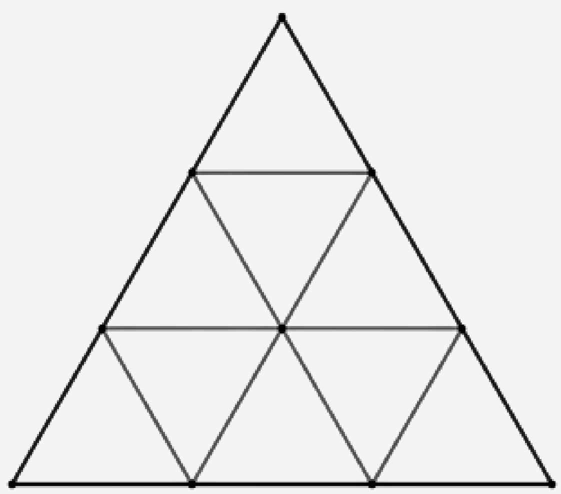
(C) 18

(D) 20

(E) 23

Solution 2.1 The first triangular shape has 1 orange. The second shape has $1+2$ or 3 oranges. The third has $1+2+3$ or 6 oranges. The fifth triangular shape has $1+2+3+4+5$ or 15 oranges.

Example 2.2 If we count triangles of any size, how many triangles are there in the following picture?



(A) 9

(B) 10

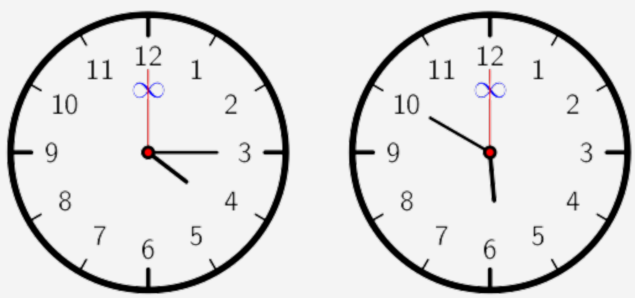
(C) 12

(D) 13

(E) 15

Solution 2.2 There are 9 little triangles, 3 medium-sized triangles, and 1 big triangle. Therefore, the total number is $1+3+9$ or 13 triangles.

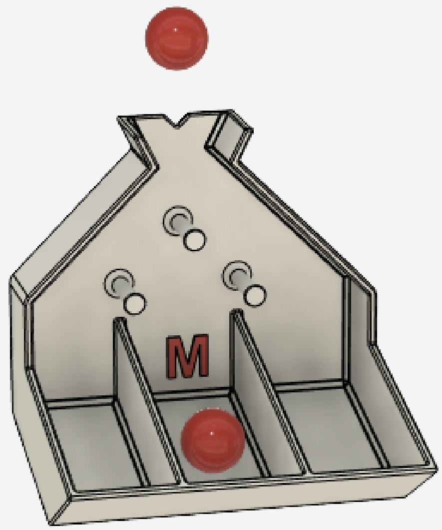
Example 2.3 Lacie started her daily reading at 4:15 pm in the afternoon and stopped reading at 5:50 pm in the evening. How many minutes did she read?



Your Answer:____________________

Solution 2.3 There are several ways to work on the problem. As an example, we could say that from 4:15 pm to 5:15 pm, Lacie read one hour or 60 minutes. Then, there are 35 minutes from 5:15 to 5:50. So the total time is 95 minutes.

Example 2.4 A marble is dropped at the top opening of the device shown in the picture. At each peg, it goes either to the left or right. In how many ways can a marble go into the bin in the middle?



(A) 2

(B) 3

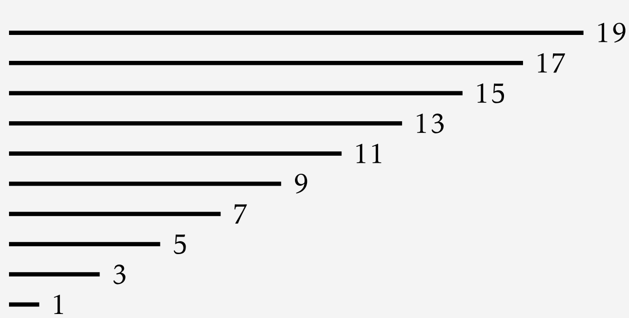
(C) 4

(D) 5

(E) 6

Solution 2.4 When a marble is dropped into the opening, it has to make two decisions. At the first peg, it goes left or right. When it goes to the left, it can go to the right at the second pin to land in the middle bin. When it goes to the right, it can go to the left at the second pin to land in the middle bin. Therefore, there are two pathways for a marble to reach the middle bin M. A drawing may help you follow the path of a marble.

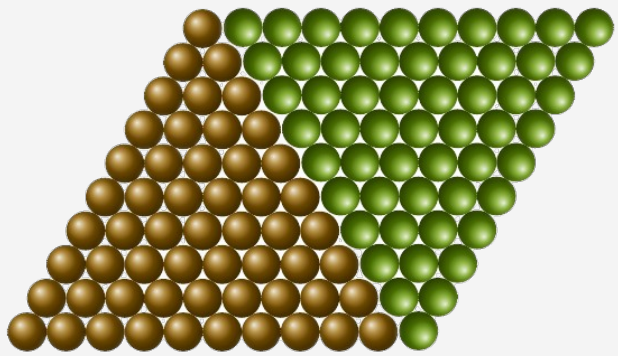
Example 2.5 Olivia has ten sticks. Their lengths are 1, 3, 5, 7, 9, 11, 13, 15, 17 19 centimeters. Olivia connects all of them from end to end for a big loop. How long is the loop in centimeters?



Your Answer:____

Solution 2.5 There are several ways to solve the problem. One way is to relate the numbers from both ends. So, we have $1+19$, $3+17$, $\ldots$, and $9+11$. Thus, there are five pairs, each of which has a sum of 20. The total is 100 centimeters.

Example 2.6 How many balls are there in the following picture?



Your Answer:____

Solution 2.6 There are several ways to count the balls. If one sees ten rows of 11 balls or eleven columns of 10 balls, the answer is just 110. One may also see two triangles. Each triangle has 55 balls. Altogether, there are 110 balls. It is certainly possible to count them one by one.

3. Grade Three

The third-grade test focuses on mathematical understanding and problem solving involving place values in the base-ten number system, number sense, multiplication and division, fractions, measurement and data, and properties of simple two-dimensional (2D) shapes. Six examples are provided with explanations.

3.1 Targeted Mathematical Ideas

- Place values and number sense in the base ten number system.
- Understanding and solving multiplication and division problems around 100.
- Understanding fractions as numbers and in context, including unit fractions.
- Solving word problems using a modeling approach.
- Measuring length, weight, liquid volume, time, and money.
- Interpreting data from a scaled picture or bar graph.
- Developing concepts of perimeter and area of plane figures.
- Pattern identification and extensions of numbers and shapes.

3.2 Examples and Solutions

Example 3.1 In the following multiplication problem, A, B, C represent different digits from 0 to 9. If $AB \times 7 = CBB$, what is the sum of A, B, and C?

$$\begin{array}{cccc} & & A & B \\ \times & & & 7 \\ \hline & C & B & B \end{array}$$

(A) 16

(B) 12

(C) 13

(D) 15

(E) 14

Solution 3.1 Consider all the multiples of 7. If $7 \times B$ leaves a unit digit of B, $B = 5$. Since $5 \times 7 = 35$, then, A must be 6 to leave $B = 5$ in the ten's place in the result. Thus $C = 4$. and the sum of A, B, and C is $5 + 6 + 4 = 15$.

Example 3.2 If we know

$$\begin{aligned} \heartsuit + \heartsuit + \heartsuit + \triangle + \triangle &= 45, \\ \triangle + \heartsuit &= 18, \end{aligned}$$

then what number is $\heartsuit$ equal to?

(A) 10

(B) 9

(C) 12

(D) 11

(E) 13

Solution 3.2 From $\triangle + \heartsuit = 18$, we know $2\triangle + 2\heartsuit = 36$. Since three hearts plus two triangles is 45, one heart $\heartsuit$ is equal to 9.

Example 3.3 Amy has 60 marbles in a box. There are red and green marbles only. The number of green marbles is three times the number of red marbles. How many more green marbles than red marbles does Amy have?

(A) 20

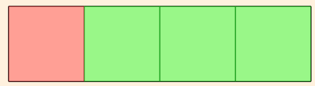
(B) 15

(C) 25

(D) 30

(E) 35

Solution 3.3 One way to solve the problem is using a model such as the one below. Each box represents one-fourth of the marbles. We need four boxes because of the information given in the problem. Green marbles account for $\frac{3}{4}$ of the total. Visually, green marbles have two more boxes than red marbles. Since each box represents 15 marbles, there are 30 more green marbles.



Example 3.4 Jenny writes down a two-digit number. If she adds a "0" to the end of this number, the new number will be 126 larger than the original number. What is this two-digit number?

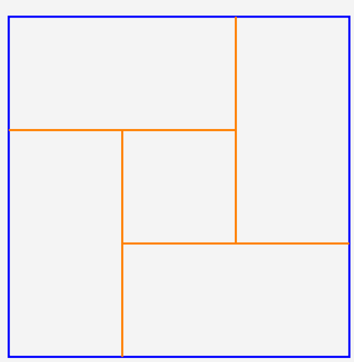
Your Answer:______________________________

Solution 3.4 While students should feel free to use algebraic ways of reasoning if they are comfortable. We can seek a pattern, starting with some examples:

- If the number was 2, then we would have $20 - 2 = 18$, which is 2×9;
- If the number was 3, then we would have $30 - 3 = 27$, which is 3×9;
- ...
- If the number was $\bigstar$, then we would have $10\bigstar - \bigstar = 9\bigstar$;

Therefore, the original two-digit number is $126 \div 9 = 14$.

Example 3.5 A big square is divided into four small rectangles of the same size and one small square, as shown in the following figure. If the perimeter of each small rectangle is 24 inches, what is the area of the big square in square inches?



Your Answer:________________________________

Solution 3.5 We need to find the side length of the big square. From observation, we can see that each side of the big square is precisely equal to the length plus the width of a small rectangle, which is one-half its perimeter. So, the side length of the big square is 12 inches and the area of the big square is $12 \times 12 = 144$ square inches.

Example 3.6 Turtle Todd is 12 meters away from a pond. He plans to get to the pond for a bath in three hours. During the first hour, he walked $\frac{1}{2}$ of the total distance. During the second hour, he walked $\frac{1}{3}$ of total distance. How far does he need to walk in the third hour?

Your Answer:________________________________

Solution 3.6 Turtle Todd walked 6 meters during the first hour and 4 meters during the second hour. There is 2 meters left for the third hour. A visual model or the number line could be helpful.

4. Grade Four

Fourth-grade students are expected to have a flexible understanding of the base ten number system and its properties and use all the four arithmetic operations to solve multi-step problems in context. Fraction equivalence, operations with fractions, algebraic thinking, and properties of two-dimensional shapes are also covered. Five examples are provided with explanations.

4.1 Targeted Mathematical Ideas

- Understanding of numbers and operations in the base ten number system.
- Four arithmetical operations on multi-digit whole numbers.
- Building upon the understanding of whole-number addition and multiplication to include fractions.
- Understanding factors, multiples, and divisibility.
- Recognizing patterns in number sequences or shapes to solve problems.
- Properties of two-dimensional shapes, including lines and angles.
- Solving real-world problems involving time, money, distance, weight, and liquid volumes.
- Creating mathematical models to solve challenging word problems.

4.2 Examples and Solutions

Example 4.1 Three children can make 3 origami paper boats in 3 minutes. If all children can work at the same rate, how many boats can 12 children make in 12 minutes?

(A) 12

(B) 24

(C) 48

(D) 36

(E) 18

Solution 4.1 We could approach the problem in two steps. At the first step, we find out how many boats can 12 children make in 3 minutes. If 3 children can make 3 boats in 3 minutes, then 12 children can make 12 boats. At the second step, we find out how many boats can 12 children make in 12 minutes. Since 12 children make 12 boats in 3 minutes, they can make four times as many or 48 boats in 12 minutes.

Example 4.2 Audrey has a box of stickers. One-fifth of the stickers are stars, two-sevenths of them are diamonds, and the rest are flowers. If there are no more than 50 stickers in the box, how many stickers does Audrey have?

(A) 30

(B) 28

(C) 35

(D) 42

(E) 40

Solution 4.2 One way to approach this problem is to use the idea of a common multiple. The total number of stickers needs to be an integer or whole number, so it must be a multiple of both 5 and 7, which is 35, 70, and so on. As there are fewer than 50 stickers in the box, Audrey can only have 35 stickers.

Example 4.3 Charlie has read one-third of a book. If he reads 100 more pages, he will have finished half of the book. How many pages does the book have?

(A) 300

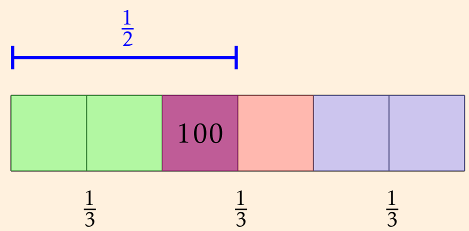
(B) 400

(C) 450

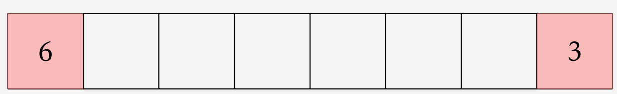
(D) 500

(E) 600

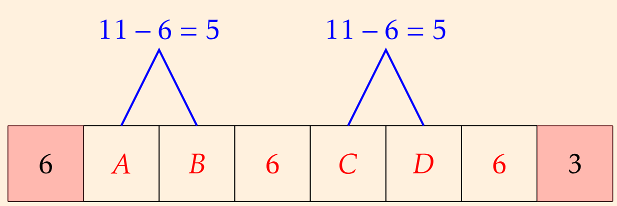
Solution 4.3 We can approach this problem using a modeling approach. From the model, we can tell 100 is $\frac{1}{2} - \frac{1}{3} = \frac{1}{6}$ of the whole book. Therefore, the book has 600 pages. Algebraic equations can certainly be used when students are comfortable with them.



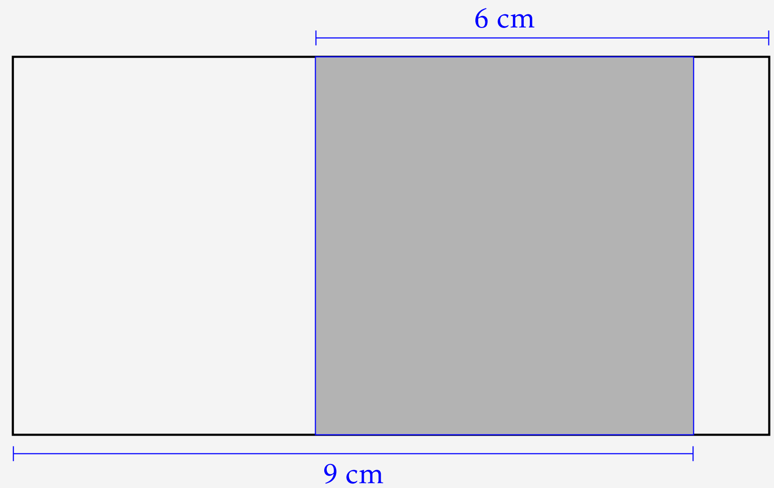
Example 4.4 The first and last digit of an 8-digit number is shown in the following figure. If the sum of any three adjacent digits is 11. What is this number?



Solution 4.4 Since any three adjacent digits add up to 11, we have the following patterns, where $A + B = 5$, and $C + D = 5$. Thus, the fourth digit must be 6, and the seventh digit must also be 6. Now, since $D+6+3 = 11$, we have $D = 2$. Hence, $C = 3$, $B = 2$ and $A = 3$. The number is 63263263.



Example 4.5 In the following figure, the shaded area is a square. All the other shapes are rectangles. What is the perimeter of the largest rectangle in the figure?



Your Answer:____

Solution 4.5 The shaded area is a square, so the sides of the square have the same length as the width of the large rectangle. Therefore, the length plus the width of the large rectangle is equal to $9 + 6 = 15$. Thus, the perimeter is 30.

5. Grade Five

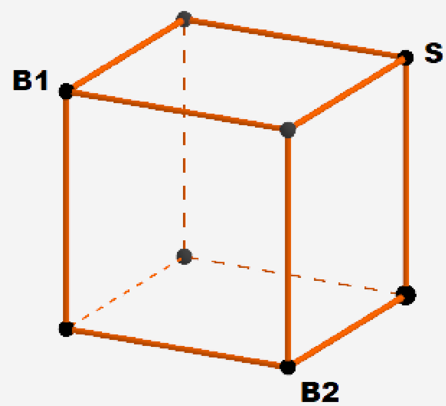
In the fifth grade, students are expected to have a solid understanding of the base-ten number system, including the meaning and operations of decimals. Fractions should be used as both ratios and numbers in a variety of problem situations. Areas of 2D shapes and volumes of 3D solids, including other geometric properties, should be approached with meaning as well as rules. The Cartesian System helps make connections between geometry and algebra. Counting strategies should be used to make rich connections among arithmetic, algebra, and elementary ideas of chance or probability. Five examples are listed below with solutions.

5.1 Targeted Mathematical Ideas

- The base-ten place value system and operations with integers and decimals to hundredths.
- Use fractions and special percentages as numbers and make sense of their operations in real-world contexts.
- Make sense of number patterns and numerical expressions.
- Use measurements in real-world situations, represent and analyze data using visuals and descriptors such as average, minimum, maximum, median, mode.
- Areas of 2D shapes and volumes of 3D solids and the underlying concepts such as unit squares and unit cubes.
- Graph points, lines, and make sense of simple relations in the Cartesian Coordinate System.
- Counting strategies and chances in real-world contexts.
- Extensions and problem solving scenarios involving one or more of the above ideas.

5.2 Examples and Solutions

Example 5.1 A spider is currently at point S on a cube shown in the figure below. There is a little bug at point $B1$; there is another bug at point $B2$. The spider needs to catch the two bugs as soon as possible. It can only walk along the edges of the cube. In how many ways can the spider catch the two bugs?



(A) 4

(B) 6

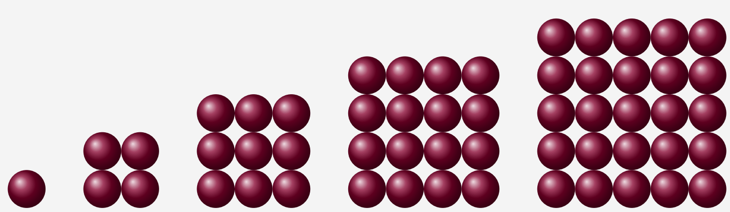
(C) 7

(D) 8

(E) 10

Solution 5.1 If the spider wants to catch $B1$ first and then $B2$, there are 4 paths along the edges. If the spider gets to $B2$ first and then $B1$, there are another four paths along the edges. So, in total, the spider has 8 choices, each of which is 4 edges long.

Example 5.2 Katie was arranging a large number of red apples into square shapes, as shown in the figure. The first shape has one apple. The second shape has four apples. The third shape has nine apples. Katie was able to complete the sixth square with all the apples she had. How many apples did Katie use in making all the square shapes?



(A) 36

(B) 61

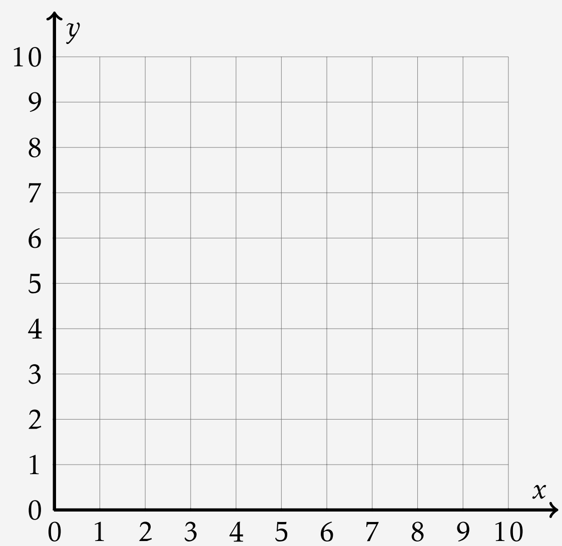
(C) 77

(D) 86

(E) 91

Solution 5.2 The sixth square has 36 apples. Therefore, the total is the sum of $36+25+16+9+4+1$, which is 91 apples.

Example 5.3 In the xy-plane, a turtle is currently at the point $(1,3)$. There is a worm at point $(8,9)$. The turtle can only move horizontally or vertically along the grid one step at a time. One step is one unit. How many steps does the turtle have to walk to get to the worm as soon as possible?



(A) 10

(B) 11

(C) 12

(D) 13

(E) 21

Solution 5.3 From 1 to 8, there are 7 horizontal steps. From 3 to 9, there are 6 vertical steps. In total, there are 13 steps.

Example 5.4 Katrina has a bag of colored marbles. She is told that two-fifths of the marbles are red, one-half are blue, and the rest are green. She found that there are 6 green marbles. How many red marbles are in the bag?

(A) 12

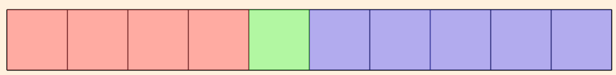
(B) 18

(C) 24

(D) 30

(E) 36

Solution 5.4 One-tenth of the marbles are green. So, one-tenth of the total number is 6. Therefore, there are 24 red marbles. A visual model may be helpful to analyze the fractional parts of the marbles.



Example 5.5 Nora is playing with numbers. She uses all the three digits in {8, 1, 5} to make a three-digit whole number. What is the difference between the largest number and the smallest number she can make?

Your Answer:____________________

Solution 5.5 The largest whole number is 851. The smallest number is 158. The difference between 851 and 158 is 693.

Example 5.6 A certain type of hand sanitizer can kill one-half of the germs after one use. If it has the same power when used repeatedly, what fraction of the germs will be killed after three applications?

Your Answer:____________________

Solution 5.6 We can model the process using fractions or percentages. After one application, $\frac{1}{2}$ of the germs are killed, which means $\frac{1}{2}$ of the germs are still alive. After the 2nd application, $\frac{1}{2}$ of the germs that are still alive are killed, leaving $\frac{1}{4}$ of the germs alive. After the third application, $\frac{1}{8}$ of the germs are still alive. Therefore, in total, $\frac{7}{8}$ of the germs are killed after three applications. Alternatively, we can draw a picture or start with a specific number, say, 16 germs, to reason about the problem situation.

6. Grade Six

In the sixth grade, students are expected to be proficient with the number system and related arithmetic operations involving integers and fractions, including ratios and proportions, algebraic expressions, and equations. Also covered are the surface area and volume of 3D geometrical objects, statistical variability, and data distribution.

6.1 Targeted Mathematical Ideas

- Understanding the base-ten number system and its properties.
- Solving arithmetic problems involving whole numbers, fractions, mixed numbers, and decimals.
- Finding factors and multiples of a whole number; understanding prime factorization.
- Understanding data variability and distribution, including related quantitative measures.
- Linear algebraic expressions and inequalities.
- Counting strategies.
- Spatial visualization of 2D and 3D shapes; solving problems involving area, surface area, and volume.
- Extensions and integration of mathematical ideas in the above areas.

6.2 Examples and Solutions

Example 6.1 Amy has a rolling wheel with a diameter 50 centimeters. She rolled it from point A to point B at a steady pace. The wheel made 800 revolutions. It took Amy 30 minutes. Consider a point P on the outer edge of the wheel. What is the speed at which point P was rotating in meters per hour?

(A) 1500π

(B) 1500π

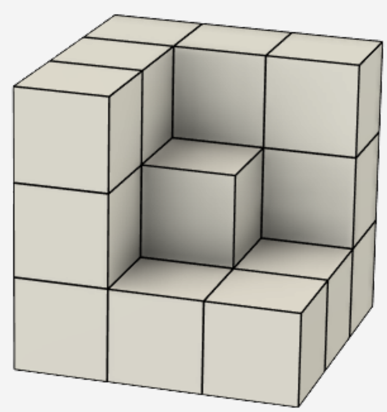
(C) 800π

(D) 2400

(E) 4000π

Solution 6.1 After 800 revolutions, the wheel or point P has covered a distance of $0.5\pi \times 800 = 400\pi$ meters. Since it took 0.5 hours, the (linear) speed of point P is 800π meters per hour.

Example 6.2 A three-cube has some unit cubes removed from its top right corner, as shown in the figures. There are no other pieces removed from other parts of the cube. Holly wants to attach a sticker to each unit square face of the solid. How many stickers does Holly need?



(A) 36

(B) 27

(C) 48

(D) 54

(E) 60

Solution 6.2 We can certainly count the number of square faces one by one or using some counting strategies. Interestingly, we could mentally push the unit square faces out and observe that it is just a whole three-cube as long as the outer surfaces are concerned. So, Holly needs 54 stickers.

Example 6.3 How many factors are there in the number $1,000,000$, including one and itself.

(A) 36

(B) 49

(C) 56

(D) 64

(E) 100

Solution 6.3 With prime factorization, we $1,000,000 = 2^6 \cdot 5^6$. To make a factor, we need to choose some number of 2 and some number of 5, each of which has 7 choices. Therefore, there are 49 factors in one million, including one and itself.

Example 6.4 Cassie is paid at a fixed hourly rate at a local bookstore with some base wage. When she works 24 hours a week, she is paid \$388 weekly. If she works 40 hours a week, she is paid \$580 weekly. How much will she make weekly if she works 30 hours a week?

(A) 435

(B) 460

(C) 480

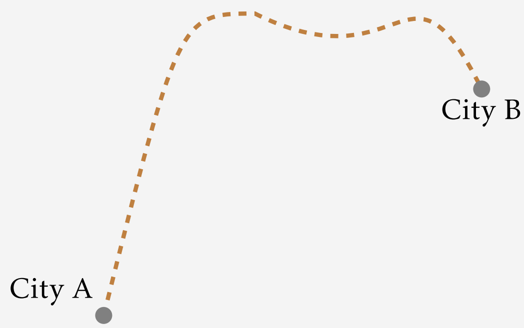
(D) 485

(E) 496

Solution 6.4 There are several ways to solve the problem. We could make a table to organize the information if necessary. Essentially, we know the hourly rate remains a constant or specifically, $(580-388) \div (40-24) = 12$. Thus, the base way is $388 - 12 \times 24 = 100$. Thus, for 30 hours, Cassie will be paid $100 + 12 \times 30 = 460$. Alternatively, we could let x represent wage for 30 hours. Then, we could set up an equation $\frac{x-388}{30-24} = \frac{580-388}{40-24}$. Solving the equation, we get $x = 460$.

Hours	Wage
24	388
30	x
40	580

Example 6.5 Hailey followed the same route from City A to City B. Her car averaged 90 kilometers per hour (kmph) on a trip to City B. It averaged 60 kmph on the way back to City A because of the bad weather conditions. What was the average speed for her round trip in kilometers per hour (kmph)?



Your Answer:____

Solution 6.5 We could use T (or simply 1) to represent the distance from City A to City B. The round trip is accordingly $2T$, and the total amount of time of the round trip is $\left(\frac{T}{90}+\frac{T}{60}\right)$ hours. Therefore, the average speed for the round trip is

$$\frac{2T}{\frac{T}{90}+\frac{T}{60}} = 72 \text{ (kmph)}.$$

Example 6.6 Zach has 500 game cards. He made a table of the price of each card and, after some computer-based data analysis, he found the median price is \$1.2 and the mean price is \$2.4. What is the total value of Zach's game cards?

Your Answer:____

Solution 6.6 The total value of Zach's cards can be found using the mean of his data: $2.4 \times 500 = 1200$ dollars.

7. Grade Seven

In the seventh grade, students are expected to develop proficiency with rational numbers and their operations and problem solving competency involving ratios and proportions, algebraic expressions, and linear equations. Also covered are the construction and properties of common 2D and 3D geometrical objects, statistical inferences, chance processes, probability models, and related counting strategies.

7.1 Targeted Mathematical Ideas

- Number systems and their properties; arithmetic operations with rational numbers.
- Proportional relationships and their applications.
- Using and solving linear equations in real-world contexts.
- Operations involving numeric and algebraic expressions and linear equations.
- Construction and properties of geometric figures such as circles, triangles, rectangle, parallelograms, and trapezoids; solving problems involving angles, area, surface area, and volume.
- Counting strategies, number sequences.
- Understanding random sampling, inferences, and probability models.
- Extensions and integrations of the mathematical ideas mentioned above.

7.2 Examples and Solutions

Example 7.1 The following table shows part of the relationship between degrees Fahrenheit ($^\circ F$) and degrees Celsius ($^\circ C$). If we use F to represent a temperature in degrees Fahrenheit, and C for the same temperature in degrees Celsius. Which of the following equations describes the same relationship?

$°F$	$°C$
68	20
86	30
104	40
122	50

(A) $C = (F + 32) \times 5 \div 9$

(B) $C = (F - 16) \times 9 \div 5$

(C) $F = C \times 5 \div 9 - 32$

(D) $F = C \times 5 \div 9 + 16$

(E) $5F = 9C + 160$

Solution 7.1 There are certainly multiple ways to find the right equation. We could check the equations one by one by substituting a (F, C) pair into the equations. Alternatively, we could graph the (F, C) pairs or calculate the rate of change between $(°F)$ and $(°C)$. For a change of $1°C$, the corresponding change on the $°F$ scale is $\frac{86-68}{30-20} = \frac{9}{5}$. Therefore, $5F = 9C + 160$ is the correct equation for the $F - C$ relationship.

Example 7.2 In the following table, each letter is mapped to a number. If the number pattern continues, what is the sum of all the numbers that correspond to the letters in the word PACIFIC?

A	B	C	D	E	$\cdots$
6	24	60	120	210	$\cdots$

(A) 5520

(B) 6897

(C) 7338

(D) 8963

(E) 7883

Solution 7.2 Let n represent the nth letter. Then, the corresponding number for the nth letter is $n(n+1)(n+2)$. The letter P is the 16th letter and is therefore mapped to 4896. The letter I corresponds to 990; the letter F is 336; the letter C is 60; the letter A is 6. All together, the sum is 7338.

Example 7.3 A motorcyclist went 75 miles in 45 minutes. On the highway, he saw a speed limit sign that reads 25 mph (miles per hour). By what percent did he go above the speed limit?

(A) 200%

(B) 250%

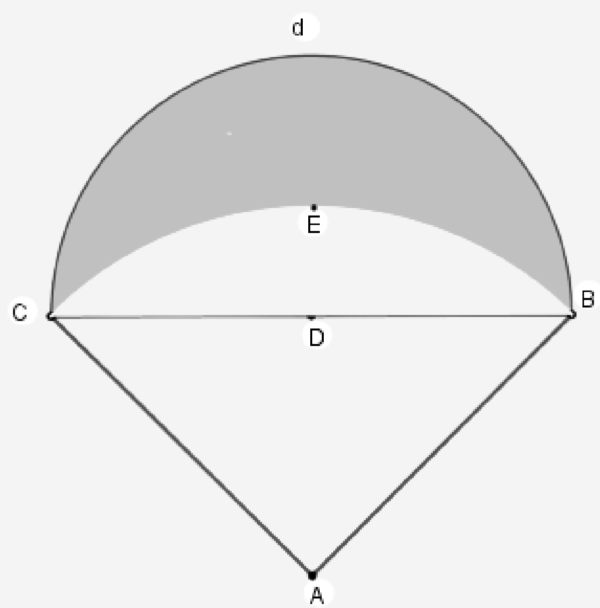
(C) 300%

(D) 320%

(E) 400%

Solution 7.3 First, we need to find the motorcyclist's current speed in miles per hour. Forty minutes is $\frac{3}{4}$ of an hour. Thus, he was riding at 100 mph. The speed limit is 25 mph. Therefore, he went above the speed limit by $\dfrac{100-25}{25} = 300\%$.

Example 7.4 In the following figure, $\angle BAC$ is a right angle. $AC = AB = 1$ centimeter. A is the center of arc BEC, D is the center of semicircle d. What is the area of the shaded crescent area?



(A) $\frac{1}{5}$

(B) $\frac{1}{4}$

(C) $\frac{1}{3}$

(D) $\frac{1}{2}$

(E) $\frac{2}{3}$

Solution 7.4 Note that $CD = \frac{\sqrt{2}}{2}$ cm. The semicircle d has an area of $\frac{\pi}{4}$ cm^2. The sector BCA has an area of $\frac{\pi}{4}$. The area of $CDBE$ is therefore $\frac{\pi}{4} - \frac{1}{2}$. Finally the shaded crescent has an area of $\frac{1}{2}$ cm^2.

Example 7.5 Jim has a huge bag of white and orange ping-pong balls. Jim knows that 60% of the balls are orange. He draws a ball from the bag in a random manner. He puts it back if it is not white. What is the probability that it will take at least five draws to get a white ball?

(A) 0.13

(B) 0.24

(C) 0.36

(D) 0.80

(E) 0.86

Solution 7.5 The probability that Jim will get an orange ball for four consecutive draws is 0.6^4, which is the same as "taking at least five draws to get a white ball." Therefore, the probability in question is $0.6^4 \approx 0.13$. Alternatively, we could analyze all the cases. If the fifth ball turns out white, the probability is $p_5 = 0.6^4 \times 0.4$; If the sixth ball is white, the probability is $p_6 = 0.6^5 \times 0.4$. If the nth ball is white, the probability is $p_5 = 0.6^{n-1} \times 0.4$. Therefore, the sum is $0.4 \times 0.6^4 \times (1 + 0.6 + 0.6^2 + 0.6^3 + \cdots)$, which is equal to 0.6^4 after summing up the infinite geometric series.

Example 7.6 Let A and B be the two solutions to the absolute value equation

$$|-4x+8|-6=18,$$

What is $|A \cdot B|$?

Your Answer:______________________________

Solution 7.6 Note that $|-4x+8| = 24$. There are two cases: $-4x+8 = 24$ and $-4x+8 = -24$. Thus, there are two solutions: $x_1 = -4$ and $x_2 = 8$. Their product is -32 and the absolute value is 32.

8. Grade Eight

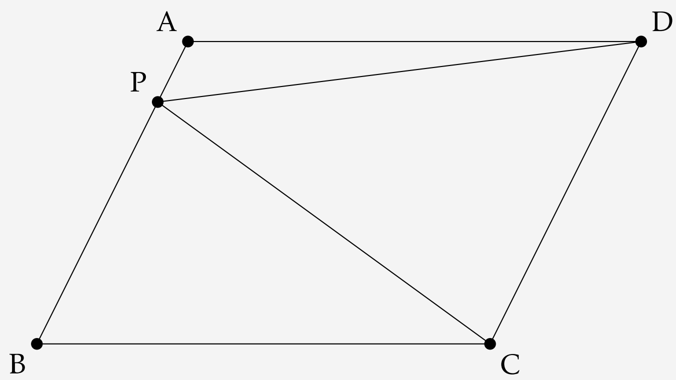
In the eighth grade, the following mathematical ideas and their extensions are targeted. Six examples are listed with explanations. There are always other ways to approach a problem. The focus is on understanding and generative mathematical reasoning in these areas.

8.1 Targeted Mathematical Ideas

- Number systems, irrational numbers, and the nature of place-value number systems.
- Radical expressions and powers with integer exponents.
- Number patterns, arithmetic sequences, algebraic expressions, linear equations, and linear functions.
- The processes and properties of similarity and congruence.
- The Pythagorean Theorem and its applications.
- Properties of 2D and 3D shapes such as triangles, quadrilaterals, cubes, cones, cylinders, and spheres.
- Counting strategies and their applications in approaching probability problems.
- Extensions and problem solving scenarios in the above areas.

8.2 Examples and Solutions

Example 8.1 Parallelogram $ABCD$ has an area of 48 square inches. P is a point on AB. What is the area of $\triangle CDP$?



(A) Not enough information

(B) 12 square inches

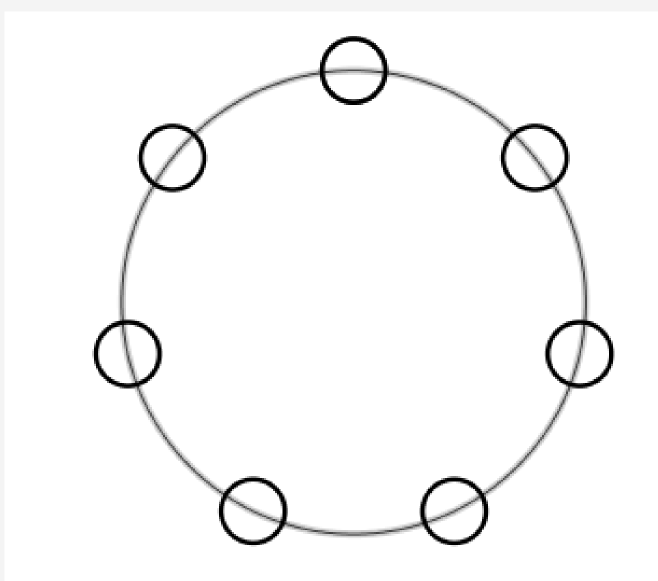
(C) 16 square inches

(D) 24 square inches

(E) $36\sqrt{5}$ square inches

Solution 8.1 The area of $\triangle CDP$ is one-half of that of the parallelogram $ABCD$. Therefore, it is 24 square inches.

Example 8.2 Mary has two colored markers, green and red. She will color all the seven little circles on a big circle. She will not flip the shape. In how many different ways can Mary color the little circles?



(A) 7

(B) 10

(C) 12

(D) 15

(E) 20

Solution 8.2 First, note that 7 is a prime number. Each little circle has two colors. Thus, there is a total of 2^7 possibilities if we ignore the rotations. There are two cases where all the little circles have the same color, either all red or all green. All the others can be rotated 7 times, including the unique one. Therefore, there is a total of $(2^7 - 2)/7 + 2 = 20$ unique patterns. There are other ways to think about the problem, depending on one's perspective such as Fermat's Little Theorem.

Example 8.3 At a local football stadium, the benches are 8 meters long. If two people sit randomly on the bench, what is the probability that they will be at least two meters apart from each other?

(A) $\frac{5}{16}$

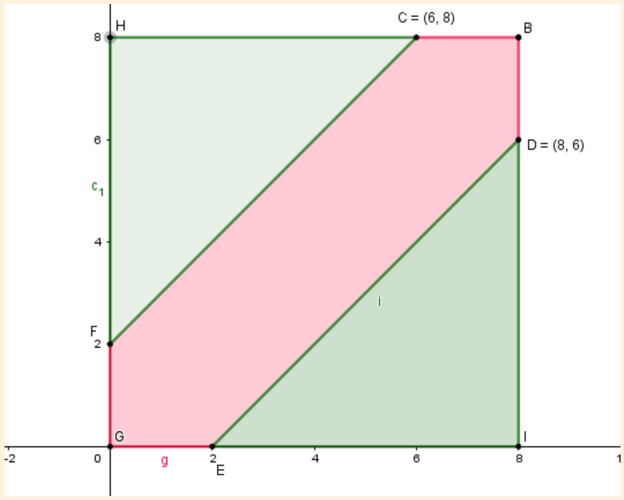
(B) $\frac{7}{16}$

(C) $\frac{9}{16}$

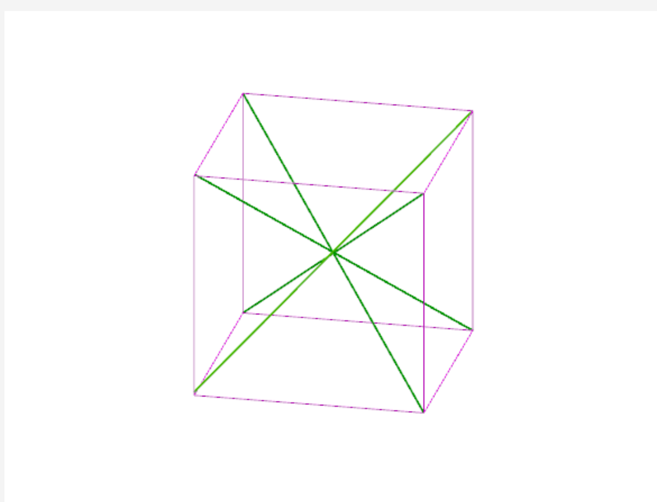
(D) $\frac{12}{16}$

(E) $\frac{15}{16}$

Solution 8.3 This is a problem involving geometric models of probability. Let x and y represent the distances of the two people from one end of the bench. Then, $0 \leq x \leq 8$ and $0 \leq y \leq 8$. All the possible pairs of (x, y) are within the square from $(0,0)$ to $(8,8)$ in the Cartesian Coordinate System. When $|x-y| > 2$, the two people are at least 2 meters apart from each other. The solution is therefore $\frac{36}{64} = \frac{9}{16}$. The details are shown in the following figure.



Example 8.4 A unit cube has 4 major diagonals connecting opposite vertices through the cube. What is the sum of the lengths of all the four major diagonals?



Your Answer:____________________________

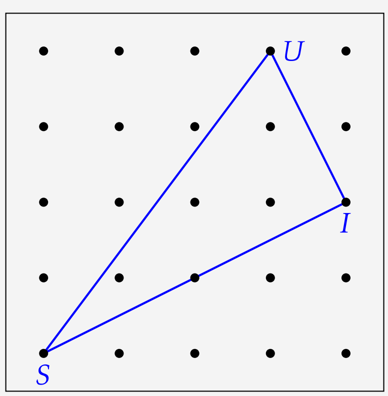
Solution 8.4 To find the length of a major diagonal, we need to use the Pythagorean Theorem two times. First, a face diagonal of a unit cube is $\sqrt{2}$ units. Then, using a face diagonal and an edge, we could find the length of one major diagonal, which is $\sqrt{3}$ units. Thus, the sum of all the four major diagonals is $4\sqrt{3}$ units.

Example 8.5 Matt has a new book in the summer. The book has 400 pages. He plans to read one page on the first day and then read four more pages every day than the previous day. In other words, he will read five pages on the second day, nine pages on the third day, and so on. How many days will it take Matt to finish the whole book?

Your Answer:____

Solution 8.5 This problem is related to the properties of arithmetic sequences. Let x be the day count. On the xth day, Matt should read $1 + 4(x - 1)$ pages. The total number of pages Matt had read by the end of the x-th day is therefore $\frac{(1+1+4(x-1))x}{2}$. We need to find the first value of x which yields $\frac{(1+1+4(x-1))x}{2} \geq 400$. It is 15.

Example 8.6 In the following grid, all the neighboring dots are *two* centimeters away from each other horizontally and vertically. Triangle SIU is marked with rubber bands. What is the area of triangle SIU in square centimeters?



Your Answer:____

Solution 8.6 There are multiple approaches to the problem. It is important to note that distance between two dots, horizontally and vertically, is two centimeters. To find the area of $\triangle SIU$, we could subtract the three right triangles around $\triangle SIU$, which gives $8^2 - 24 - 4 - 16 = 20$. Alternatively, we could use the Pythagorean Theorem to find $IU^2 = 20$, $SI^2 = 80$, and $SU^2 = 100$, which means $\triangle SIU$ is a right triangle. Its area is thus $\frac{1}{2} SI \cdot IU = 20$ square centimeters.

9. Grade Nine

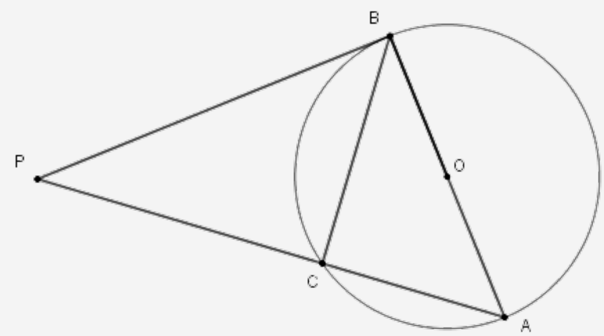
The ninth-grade test covers a variety of algebraic and geometric ideas, focusing on flexible understanding of number systems, algebraic expressions, polynomial functions, exponential functions, geometric transformations, and their applications in problem solving. Strategic counting, data analysis, and conditional probabilities are also covered. Six examples are listed below with explanations.

9.1 Targeted Mathematical Ideas

- Number systems including place-value systems other than ten; irrational numbers, radical expressions, and definitions of π, e, i, and complex numbers; prime numbers, GCD, LCM, and relatively prime numbers.
- Arithmetic and geometric sequences; linear functions, quadratic functions, exponential function, and their inverses; function graphs; function composition.
- Polynomial arithmetic, factorization, zeros of polynomials; rational expressions and radical expressions.
- Reasoning with equations and inequalities algebraically and geometrically.
- Representing and solving simple systems of equations using matrices; vectors and their operations.
- Properties of common 2D shapes such as triangles, quadrilaterals, and circles.
- Angles related to circles.
- Similarity, congruence and related processes and processes such as dilation, reflection, and rotations.
- The Pythagorean Theorem and its applications in connecting geometry and algebra in the Cartesian System.
- Counting strategies, conditional probability, data analysis, and modeling in real-world contexts.
- Extensions and problem solving scenarios involving one or more of the above ideas.

9.2 Examples and Solutions

Example 9.1 In the following figure, PB is tangent to the circle centered at O. AB is a diameter of the circle. C is on the circle. AC is extended and intersects PB at P. If $PC = 4$ inches and $AC = 3$ inches, what is the length of AB in inches?



(A) 5

(B) 7

(C) $\sqrt{21}$

(D) $\sqrt{29}$

(E) 8

Solution 9.1 There are several ways to approach the problem. As an example, we first note that $AB \perp PB$ and $BC \perp AP$. Further, $BC^2 = PC \cdot AC = 12$ through similar right triangles ($\triangle BCP \sim \triangle ACB$). Now, $AB^2 = BC^2 + AC^2$. Thus, $AB = \sqrt{21}$ inches.

Example 9.2 What is the value of the following expression?

$$\lim_{n\to\infty}\left(\sum_{i=1}^{n}\frac{1}{7^i}\right)$$

(A) $L = \frac{1}{3}$

(B) $L = \frac{1}{4}$

(C) $L = \frac{1}{5}$

(D) $L = \frac{1}{6}$

(E) $L = \frac{1}{8}$

Solution 9.2 Note that $\frac{1}{7} < 1$. Let $L = \frac{1}{7} + \frac{1}{7^2} + \frac{1}{7^3} + \cdots$. Then, $7L = 1 + \frac{1}{7} + \frac{1}{7^2} + \frac{1}{7^2} + \cdots$. Subtracting the previous two equations, we get $6L = 1$. Thus, $L = \frac{1}{6}$.

Example 9.3 Let $X = A + B + C + D$. If A, B, C, D are each randomly assigned a number from $\{2, 3\}$, what is the probability that $X = 10$?

(A) $\frac{1}{8}$

(B) $\frac{3}{16}$

(C) $\frac{1}{4}$

(D) $\frac{5}{16}$

(E) $\frac{3}{8}$

Solution 9.3 One way to approach the problem is to consider the coefficient of x^{10} in the expanded form of $(x^2 + x^3)^4$, which is 6. Thus, the probability that $X = 10$ is $\frac{6}{16}$ or $\frac{3}{8}$.

Example 9.4 Define two functions $f(x) = 3^x - 5$ and $g(x) = 5x + 4$ over real numbers. What is the value of the following expression?

$$f^{-1}\Big(g(14) + 2\Big)$$

(A) 4

(B) 5

(C) 6

(D) $\frac{1}{4}$

(E) 36

Solution 9.4 According to the function definitions, $g(14) = 74$. We need to find $w = f^{-1}(74 + 2) = f^{-1}(76)$, which means, for some w, we have $f(w) = 76$? If $f(w) = 76$, then $3^w = 81$. Therefore, $w = 4$.

Example 9.5 Two players, Beth and Carla, are playing five rounds of a chess game. The first player to win three rounds wins the whole game and takes the award. Based on the history of the players'performance. Beth has a probability of 0.6 winning a round. Carla has a probability of 0.4 winning a round. What is the probability, rounded to the second decimal place, that Beth wins the chess game of five rounds?

Your Answer: ______________________________

Solution 9.5 Beth wins in three scenarios. If the game ends after three rounds, $P_{\text{Beth wins}} = 0.6^3$. If the game ends after four rounds, Beth wins two of the first three rounds plus the 4th one, and Carla wins one round. $P_{\text{Beth wins}} = 3 \times 0.6^3 \times 0.4$. If the game ends in five rounds, Beth wins two of the first four rounds plus the 5th one and Carla wins two of the first four rounds. $P_{\text{Beth wins}} = 6 \times 0.6^3 \times 0.4^2$. All cases considered, $P_{\text{Beth wins}} \approx 0.68$.

Example 9.6 If the following equation is true, what is the product of M and N?

$$\frac{116 - 43x}{5x^2 + 36x - 48} = \frac{M}{5x - 6} - \frac{2N}{x + 8}$$

Your Answer: ______________________________

Solution 9.6 Simplify the right-hand side and then match the coefficients of the terms in the numerators. We have

$$\begin{cases} M & - & 10N & = & -43 \\ 8M & + & 12N & = & 116 \end{cases}$$

Solving the system, we get $M = 7$, $N = 5$. Therefore, $MN = 35$.

10. Grade Ten

In the tenth grade, students are expected to have a strong knowledge of algebra, geometry, and their interconnections and applications. Key ideas include number systems, complex numbers, quadratic equations, polynomial expressions, and functional relations, conditional probability, trigonometric functions, and similarity. Five examples are listed with solutions.

10.1 Targeted Mathematical Ideas

- Real and complex number systems, vectors, and their representations.
- Expansion and factorization of algebraic expressions; solving word problems using algebra, percentages, rates, money, distance-speed-time relations.
- Exponential and logarithmic functions and their properties.
- Rational, irrational, complex number operations as well as prime factorization, GCD, LCM.
- Similarity and congruence of triangles, properties of angles in a circle.
- Graphs of linear, quadratic, and polynomial functions.
- Arithmetic and geometric sequences and the related processes.
- The Pythagorean Theorem and its extensions and applications with trigonometric functions.
- Counting strategies, probability of single events, multiple events, and conditional probability.
- Trigonometric functions, trigonometric identities, and their applications in real-world contexts.
- Extension and integration of mathematical ideas in the above areas.

10.2 Examples and Solutions

Example 10.1 Define $f(x) = \dfrac{x}{4}$ and $g(x) = x^3$. What is the value of $g\big(f(2)\big) \cdot f\big(g(2)\big)$?

(A) $\frac{1}{2}$

(B) $\frac{1}{4}$

(C) $\frac{1}{8}$

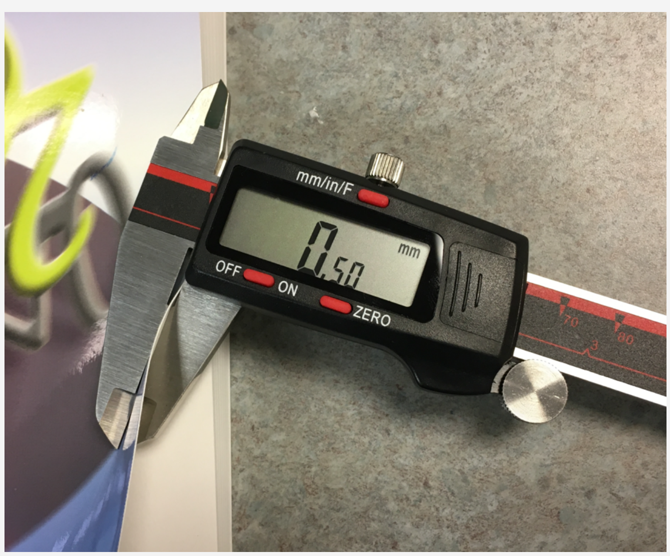
(D) $\frac{1}{10}$

(E) $\frac{1}{16}$

Solution 10.1 Given the function definitions, we have $f(2) = \dfrac{1}{2}$, and $g(2) = 8$. Using these values as inputs into g and f, respectively, we have

$$g\Big(\frac{1}{2}\Big) \cdot f(8) = \frac{1}{8} \cdot 2 = \frac{1}{4}.$$

Example 10.2 The front cover of Linda's mathematics textbook measures 0.50 mm (millimeter) thick. Linda poses a thought experiment, where she would fold the cover in half numerous times. If she folded it in half once, it would be 1mm thick. If she folded two times, it would be 2 mm thick. If she folded three times, it would be 4 mm thick. If she was able to continue, how many times would she need to fold the cover in half so that the cover would be more than 1 kilometer thick? Although logarithms are not necessary, $\log_2^{10} \approx 3.322$, if it is helpful.



(A) 16

(B) 18

(C) 20

(D) 21

(E) 100

Solution 10.2 Note that the thickness of the cover after being folded in half n times would be $t(n) = 2^{n-1}$ mm. And 1 kilometer = 1,000,000mm. Therefore, we need to find the smallest n such that $2^{n-1} \geq 1,000,000$. To find such a value for n, we could use the fact that $2^{10} = 1024$, which means $n-1 = 20$ for $t(n) \geq 1,000,000$. Thus, $n = 21$. Alternatively, we could indeed use logarithms, $n-1 = \log_2^{10^6} = 6\log_2^{10} \approx 6 \times 3.322 = 19.932$. Therefore, n should be 21, where the cover would be slightly over 1 kilometer thick.

Example 10.3 In a class of 100 students, 55 are girls and 45 are boys. They are either science or mathematics aspirants. Thirty boys are interested in a career in science, and 45 girls are interested in a career in mathematics. If we know that a student is interested in science, what is the probability that the student is a girl?

(A) 0.15

(B) 0.25

(C) 0.35

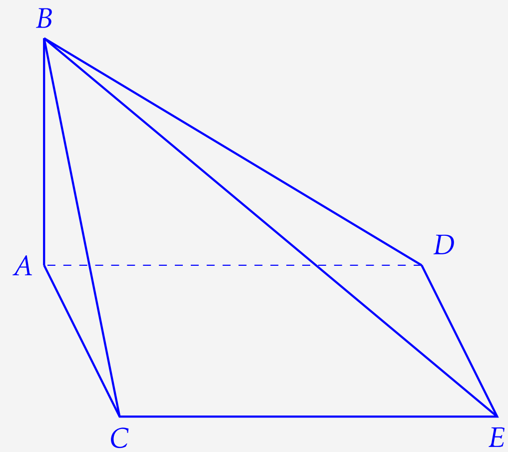
(D) 0.45

(E) 0.50

Solution 10.3 This is a problem involving conditional probability. A table helps organize the data and then solve the problem. As shown below, there are 40 students who are science aspirants, of whom 10 are girls. Therefore, the probability that the student is a girl is $10 \div 40 = 0.25$.

Career Interest	Boys	Girls	Total
Science	30	10	40
Math	15	45	60
Total	45	55	100

Example 10.4 In the following figure, $ACED$ is a rectangle. $BA \perp AC$ and $BA \perp AD$. $AB = 40$ cm. $m\angle ABC = 30°$, $m\angle ABD = 60°$, what is the distance between C and D in cm?



(A) $\frac{40\sqrt{30}}{3}$

(B) $\frac{50\sqrt{10}}{3}$

(C) $\frac{40\sqrt{10}}{3}$

(D) $\frac{20\sqrt{26}}{3}$

(E) 60

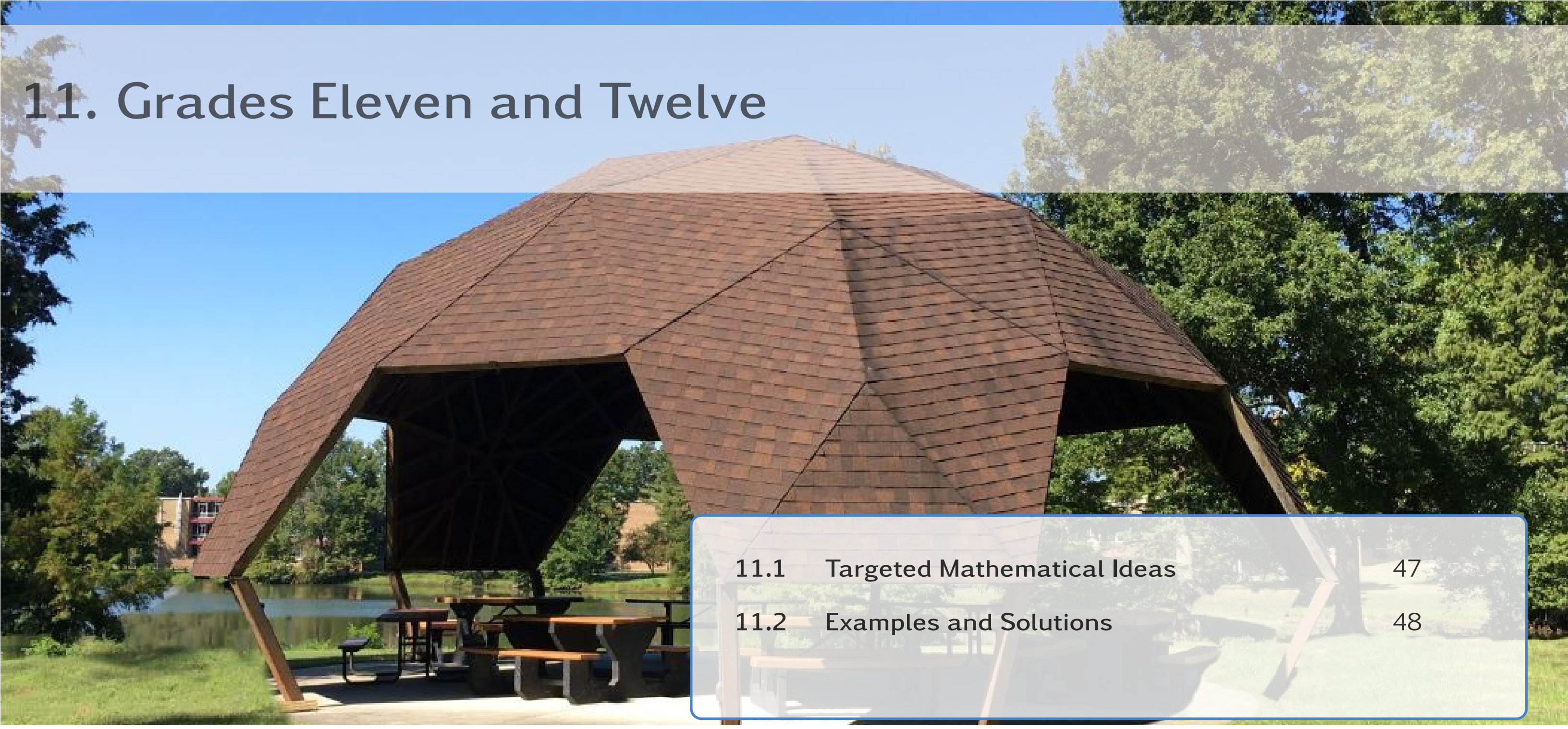
Solution 10.4 Note that $AC = AB \cdot \tan(\angle ABC)$ and $AD = AB \cdot \tan(\angle ABD)$ and $AC \perp AD$. Therefore, $CD^2 = AC^2 + AD^2 = AB^2\left(\tan^2(30°) + \tan^2(60°)\right)$, and $CD = \dfrac{40\sqrt{30}}{3}$ cm.

Example 10.5 In how many ways can you arrange the letters in the word $ABBREVIATION$ if all vowels occur together and all letters are in uppercase?

Your Answer:________________________________

Solution 10.5 There are 6 vowels and 6 consonants in the word ABBREVIATION. Letters A, I, and B each occur twice. We could treat the 6 vowels together as one letter. The total number of arrangements would be $\dfrac{7!}{2!}$ ways, where B occurs twice. The vowels can be arranged in $\dfrac{6!}{2!2!}$ ways, where A and I each occur twice. Multiplying these two, we get the number of ways we can rearrange this 12-letter word with all vowels together: $\dfrac{7!6!}{2!2!2!} = 453,600$. There are certainly other perspectives on the problem.

11. Grades Eleven and Twelve



In the 11th and 12th grades [1], students are expected to develop a comprehensive understanding of number systems, algebra, geometry, and their connections with data analysis and probability, including algebraic structures, linear and quadratic equations, inequalities, and functions. Topics in number theory, probability, geometry, sequences and trigonometry are also covered. Five examples are provided with explanations.

11.1 Targeted Mathematical Ideas

- Number systems including integers, rational numbers, real numbers, and complex numbers; rational exponents; vectors, and matrix quantities.
- Arithmetic operations on polynomials and rational expressions; solving linear and quadratic equations, inequalities, and systems of equations.
- Constructing, transforming, modeling, and graphing linear functions, quadratics functions, exponential functions, logarithm functions, rational functions, and trigonometric functions.
- Arithmetic and geometric sequences.
- Properties of triangles, rectangles, parallelograms, trapezoids, regular polygons, and circles.
- Properties and processes of perpendicular bisectors of segments and angle bisectors.
- Concepts of congruence, similarity, symmetry, and related geometry transformations such as translations, rotations, and reflection.
- Algebra and geometry connections in the Cartesian Coordinate System.
- Counting techniques, independence, conditional probability, and rules of probability.
- Extensions and problem solving scenarios in the above areas.

[1] Grades 11 and 12 use the same test form.

11.2 Examples and Solutions

Example 11.1 If $\tan\theta = -2$, what is the value of the following expression?

$$\frac{\sin\theta(1+\sin 2\theta)}{\sin\theta+\cos\theta}$$

(A) $-\frac{4}{5}$

(B) $-\frac{2}{5}$

(C) $\frac{2}{5}$

(D) $\frac{4}{5}$

(E) $\frac{6}{5}$

Solution 11.1 To solve the problem, two trigonometric identities need to be used: $\sin 2\theta = 2\sin\theta\cos\theta$, and $\sin^2\theta + \cos^2\theta = 1$. Note that $1+\sin 2\theta = \sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta = (\sin\theta+\cos\theta)^2$. Therefore,

$$\frac{\sin\theta(1+\sin 2\theta)}{\sin\theta+\cos\theta} = \frac{\sin\theta(\sin\theta+\cos\theta)^2}{\sin\theta+\cos\theta} = \frac{\sin\theta(\sin\theta+\cos\theta)}{\sin^2\theta+\cos^2\theta} = \frac{\tan\theta(\tan\theta+1)}{\tan^2\theta+1} = \frac{2}{5}.$$

Alternatively, we can start with $\tan\theta = -2$ and obtain $\sin\theta = \pm\frac{2}{\sqrt{5}}$ and $\cos\theta = \mp\frac{1}{\sqrt{5}}$ and then find the value of the expression.

Example 11.2 If $x-1 = \frac{y-2}{2} = \frac{3-z}{3}$, and $x^2+y^2+z^2$ takes its smallest value when $(x,y,z) = (a,b,c)$, what is $a+b+c$?

(A) -4

(B) -2

(C) 0

(D) 4

(E) 6

Solution 11.2 Let $x-1=\dfrac{y-2}{2}=\dfrac{3-z}{3}=k$. Then $x=k+1$, $y=2k+2$, $z=3-3k$. It follows that

$$\begin{aligned} x^2+y^2+z^2 &= (k+1)^2+4(k+1)^2+9(k-1)^2 \\ &= 14k^2-8k+14 \\ &= 14\left(k-\frac{2}{7}\right)^2-\frac{8}{7}+14, \end{aligned}$$

which has a minimum of $\dfrac{90}{7}$ at $k=\dfrac{2}{7}$. Thus, $x^2+y^2+z^2$ reaches its minimun when $k=\dfrac{2}{7}$, and

$$a+b+c=\frac{2}{7}+1+\frac{4}{7}+2+3-\frac{6}{7}=6.$$

Example 11.3 Find $f(\csc^2\theta)$ if $f\left(\dfrac{x+1}{x-1}\right)=\dfrac{1}{2}x-\dfrac{1}{2}$ for all $x\neq 1$ and $\theta\in\left(0,\dfrac{\pi}{2}\right)$.

(A) $\sin^2\theta$

(B) $\cos^2\theta$

(C) $\frac{2}{\sqrt{2}}$

(D) $\sec^2\theta$

(E) $\tan^2\theta$

Solution 11.3 First of all, use substitution to find the function expression. Next, obtain the desired function output. Let $t=\dfrac{x+1}{x-1}$. Then $x=\dfrac{t+1}{t-1}$. Substituting into the given function f, we get $f(t)=\dfrac{1}{t-1}$. Hence $f(\csc^2\theta)=\dfrac{1}{\csc^2\theta-1}=\dfrac{\sin^2\theta}{1-\sin^2\theta}=\dfrac{\sin^2\theta}{\cos^2\theta}=\tan^2\theta$.

Example 11.4 An ant is sitting at $(0,0)$ on an infinitely large coordinate plane. The ant begins a sequence of moves, each with length 1 and parallel to the horizontal or vertical axes. The direction of the move is chosen at random. The sequence ends when the ant reaches a side of the square defined by vertices $(-1,2)$, $(-1,-2)$, $(3,2)$, and $(3,-2)$. What is the probability that the ant stops at a horizontal side.

(A) $\frac{1}{4}$

(B) $\frac{3}{4}$

(C) $\frac{3}{8}$

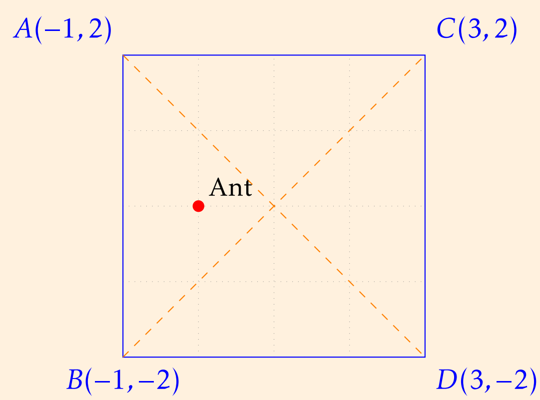
(D) $\frac{5}{8}$

(E) $\frac{2}{5}$

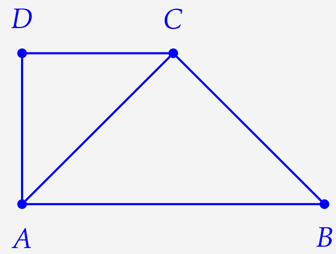
Solution 11.4 We can use the complementary counting strategy to find the probability of the ant hitting a horizontal side by finding the probability of the ant hitting a vertical side and then subtracting it from 1.

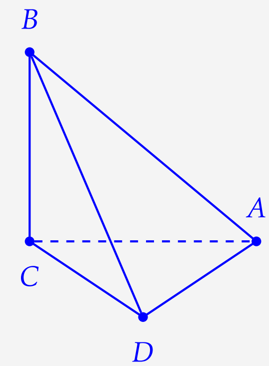
Drawing out the square, we can see that if the ant goes left, it will hit a vertical side immediately. Therefore, the probability of this happening is $\dfrac{1}{4} \cdot 1 = \dfrac{1}{4}$. If the ant goes right, it will arrive at the center of the square, and by symmetry, since the ant is equidistant to all the sides, the chance of it hitting a vertical wall is $\dfrac{1}{4} \cdot \dfrac{1}{2} = \dfrac{1}{8}$.

Now, if the ant goes either up or down, it will hit one of the two diagonals. Again, it is equidistant from the two closer sides and also the two farther sides. The probability of the ant hitting a vertical side in these two cases is $2 \cdot \dfrac{1}{4} \cdot \dfrac{1}{2} = \dfrac{1}{4}$. Summing up all the cases and subtracting the result from one, we have $1 - \left(\dfrac{1}{4} + \dfrac{1}{8} + \dfrac{1}{4}\right) = \frac{3}{8}$ as the probability that ant hits a horizontal side.



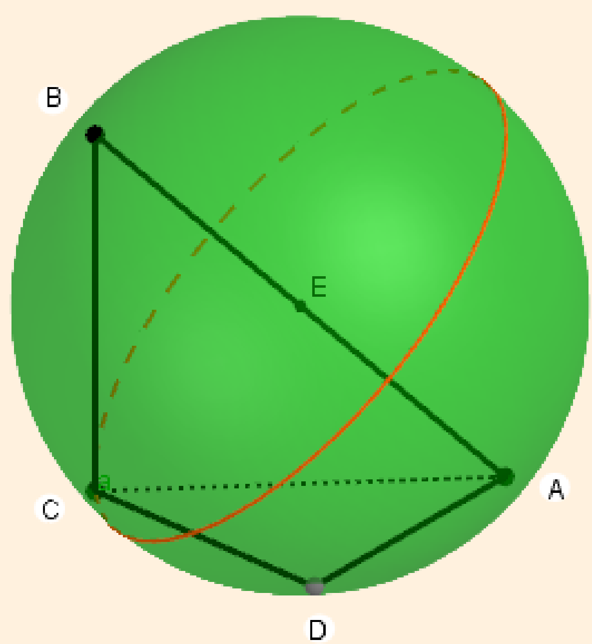
Example 11.5 In trapezoid $ABCD$, $AB \parallel CD$, $m(\angle CDA) = \frac{\pi}{2}$, $AB = 4$ units, $AD = CD = 2$ units. Along the diagonal AC, fold the trapezoid into a triangular pyramid so that $BD = 2\sqrt{3}$ units. A circumscribed sphere is constructed for the pyramid that touches all its vertices A, B, C, and D. What is the radius of the circumscribed sphere of the pyramid in units?





Your Answer:____

Solution 11.5 Note that $BD^2 = BC^2 + CD^2$, since $AC = BC = 2\sqrt{2}$. $\triangle BCD$ is a right triangle. By the same reasoning, one can see $\triangle ACB$ and $\triangle ADB$ are also right triangles. We need a point that is equally distant to all the vertices. Therefore, it is the midpoint of AB and its distance (the radius of the sphere) 2 units.



Your Notes

Important Dates

American
Mathematics Olympiad

Mathematical Thoughts

American
Mathematics Olympiad